THEORY FOR PLATES OF MEDIUM THICKNESS

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A theory of elastic isotropic plates of constant thickness is constructed without assumptions about the nature of the deformations of the transverse linear elements.

The stresses σ_{xx}, σ_{xy}, σ_{yy} are expanded in a series of Legendre polynomials $P_k\ (2z/h)$. The remaining stresses are found from equilibrium equations after application of Castigliano's principle.

The expansion of unknown quantities in Legendre polynomials was applied to the shell theory by Cicala [1]. But he made use of the principle of virtual displacements, which does not reveal all the advantages of these series over power series.

Application of the Castigliano principle gives the possibility of using these advantages effectively with the result that expansion in Legendre polynomials permits a separation of those parts of the stress for which the principal vector and moment is equal to zero. Because of this circumstance the boundary condition is formulated in the most convenient form for application of the St. Venant principle.

By neglect of terms of the order of $(h/a)^2$ by comparison with unity (h, plate thickness, a, plate width) one may obtain the equations of classical plate theory from those of the present theory. Conservation of these terms leads to exact equations which contain other terms besides those in [2-4].

1. The middle surface of the plate is described by a Cartesian rectangular xyz coordinate system. Besides the xyz coordinates we introduce the nondimensional coordinates

$$\xi = \frac{x}{a}, \qquad \eta = \frac{y}{a}, \qquad \zeta = \frac{2z}{h} \quad (-1 \leqslant \zeta \leqslant 1)$$

The stresses σ_{xx}, σ_{xy}, σ_{yy} in the plate are represented in the form of Legendre polynomials in the coordinate ζ

$$\sigma_{xx} = \frac{T_{xx}}{h} + \frac{6\zeta}{h^2} M_{xx} + \sum_{k=2}^{\infty} P_k(\zeta)\,\sigma_{kxx} \qquad (xy)$$
$$\sigma_{xy} = \frac{T_{xy}}{h} + \frac{6\zeta}{h^2} M_{xy} + \sum_{k=2}^{\infty} P_k(\zeta)\,\sigma_{kxy} \tag{1.1}$$

Here and from now on the symbol (xy) signifies that analogous relations for other quantities are obtained by the interchange of x and y.

By virtue of the orthogonality of the Legendre polynomials, T_{xx}, ..., M_{yy} have the significance of forces and moments, and $P_k(\zeta)\sigma_{kxx}$, ..., self-equilibrating stresses through the plate thickness.

For simplicity we consider the surfaces $z = \pm h/z$ of the plate to be loaded by only continuously distributed normal loads

$$\sigma_{zz} = \begin{cases} (p+q)/2 & \text{for } z = h/2 \\ (q-p)/2 & \text{for } z = -h/2 \end{cases}$$
$$\sigma_{xz} = \sigma_{yz} = 0 \qquad \text{for } z = \pm h/2 \tag{1.2}$$

Substitution of the Expressions (1.1) into the equilibrium equations of the theory of elasticity and integration with respect to z using (1.2) gives expressions for the remaining stress components (mass forces omitted)

$$\sigma_{xz} = \frac{3}{2h}(1-\zeta^2)V_x + \frac{h}{2a}\sum_{k=2}^{\infty} \frac{P_{k-1}(\zeta) - P_{k+1}(\zeta)}{2k+1} A_{kx} \qquad (xy) \tag{1.3}$$
$$\sigma_{zz} = \frac{q}{2} + \frac{3\zeta}{4}\left(1 - \frac{\zeta^2}{3}\right)p +$$
$$+ \frac{h^2}{4a^2}\sum_{k=2}^{\infty}\left[\frac{P_{k-2}(\zeta)}{(2k-1)(2k+1)} - \frac{2P_k(\zeta)}{(2k-1)(2k+3)} + \frac{P_{k+2}(\zeta)}{(2k+1)(2k+3)}\right] B_k$$

and the equilibrium equations for forces and moments

$$\frac{\partial T_{xx}}{\partial \xi} + \frac{\partial T_{xy}}{\partial \eta} = 0, \qquad \frac{\partial V_x}{\partial \xi} + \frac{\partial V_y}{\partial \eta} + ap = 0 \qquad (xy)$$
$$\frac{\partial M_{xx}}{\partial \xi} + \frac{\partial M_{xy}}{\partial \eta} - aV_x = 0 \qquad (xy) \tag{1.4}$$

Here

$$A_{kx} = \frac{\partial \sigma_{kxx}}{\partial \xi} + \frac{\partial \sigma_{kxy}}{\partial \eta}, \qquad B_k = \frac{\partial A_{kx}}{\partial \xi} + \frac{\partial A_{ky}}{\partial \eta} \qquad (xy) \tag{1.5}$$

The quantities V_x and V_y represent shear forces and A_{kx}, A_{ky} determine self-equilibrating shear forces through the thickness.

It follows that Expressions (1.1) and (1.3) satisfy the equilibrium equations of the theory of elasticity and the boundary conditions (1.2), if forces and moments are introduced therein satisfying the equilibrium equations (1.4) of plate theory.

For determination of the quantities T_{xx}, ..., M_{xx}, ..., σ_{kxx}, ..., we make use of the Castigliano principle; the problem leads to an extremum condition since the forces and moments must be subject to Equations (1.4).

As usual, we use the method of Lagrangean undetermined multipliers. We insert the left-hand sides of Equations (1.4) inside the integral for potential energy of the plate, multiplying each by an undetermined multiplier. As a result we obtain the functional

$$\Pi = \frac{h}{4E}\iint\Big\{\int_{-1}^{1}[\sigma_{xx}^2+\sigma_{yy}^2+\sigma_{zz}^2-2\nu(\sigma_{xx}\sigma_{yy}+\sigma_{xx}\sigma_{zz}+\sigma_{yy}\sigma_{zz})+ + 2(1+\nu)(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{xz}^2)]\,d\zeta + 2E\frac{u}{h}\left(\frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}\right)+ + 2E\frac{v}{h}\left(\frac{\partial T_{xy}}{\partial x}+\frac{\partial T_{yy}}{\partial y}\right)+2E\frac{w}{h}\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+p\right)+ + 2E\frac{\varphi_x}{h}\left(\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}-V_x\right)+2E\frac{\varphi_y}{h}\left(\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}-V_y\right)\Big\}\,dx\,dy \tag{1.6}$$

Here u, v, w, φ_x, φ_y are Lagrangean multipliers. The double integral is taken in the middle surface of the plate.

Formulas (1.1) and (1.3) for stresses are inserted into (1.6) and the variation of the functional is equated to zero. The variational equation will result in relations which must be satisfied in the middle surface of the plate.

If only the first four terms are retained in the series (1.1) ($\sigma_{kxx} = \sigma_{kyy} = \sigma_{kxy} = 0$, $k \geqslant 4$), these equations take the following form

$$\frac{T_{xx}}{h} = \frac{E}{(1-\nu^2)a}\left(\frac{\partial u}{\partial \xi}+\nu\frac{\partial v}{\partial \eta}\right)+\frac{\nu}{2(1-\nu)}q+\left\{\frac{\nu}{60(1-\nu)}\left(\frac{h}{a}\right)^2 B_2\right\} \qquad (xy) \tag{1.7}$$

$$\frac{T_{xy}}{h} = \frac{E}{2(1+\nu)a}\left(\frac{\partial u}{\partial \eta}+\frac{\partial v}{\partial \xi}\right)$$

$$\sigma_{2xx} = \frac{\nu h}{12(1-\nu^2)a^2}\left[\frac{\partial^2(T_{xx}+T_{yy})}{\partial \xi^2}+\nu\frac{\partial^2(T_{xx}+T_{yy})}{\partial \eta^2}\right]- - \frac{1}{24(1-\nu^2)}\left(\frac{h}{a}\right)^2\left(\frac{\partial^2 q}{\partial \xi^2}+\nu\frac{\partial^2 q}{\partial \eta^2}\right)+$$

$$+\frac{1}{21(1-\nu)}\left(\frac{h}{a}\right)^2\left(\frac{\partial A_{2x}}{\partial \xi}+\nu\frac{\partial A_{2y}}{\partial \eta}\right)-\frac{\nu}{42(1-\nu)}\left(\frac{h}{a}\right)^2 B_2-$$
$$-\frac{\nu}{42(1-\nu^2)}\left(\frac{h}{a}\right)^2\left[\frac{\partial^2(\sigma_{2xx}+\sigma_{2yy})}{\partial \xi^2}+\nu\frac{\partial^2(\sigma_{2xx}+\sigma_{2yy})}{\partial \eta^2}\right]-$$
$$-\frac{1}{1008(1-\nu^2)}\left(\frac{h}{a}\right)^4\left(\frac{\partial^2 B_2}{\partial \xi^2}+\nu\frac{\partial^2 B_2}{\partial \eta^2}\right) \quad (xy) \qquad (1.8)$$

$$\sigma_{2xy}=\frac{\nu h}{12(1+\nu)a^2}\frac{\partial^2(T_{xx}+T_{yy})}{\partial \xi\,\partial \eta}-\frac{1}{24(1+\nu)}\left(\frac{h}{a}\right)^2\frac{\partial^2 q}{\partial \xi \partial \eta}+$$
$$+\frac{1}{42}\left(\frac{h}{a}\right)^2\left(\frac{\partial A_{2x}}{\partial \eta}+\frac{\partial A_{2y}}{\partial \xi}\right)-\frac{\nu}{42(1+\nu)}\left(\frac{h}{a}\right)^2\frac{\partial^2(\sigma_{2xx}+\sigma_{2yy})}{\partial \xi\,\partial \eta}-$$
$$-\frac{1}{1008(1+\nu)}\left(\frac{h}{a}\right)^4\frac{\partial^2 B_2}{\partial \xi\,\partial \eta}$$

$$\frac{6M_{xx}}{h^2}=-\frac{Eh}{2(1-\nu^2)a^2}\left(\frac{\partial^2 w}{\partial \xi^2}+\nu\frac{\partial^2 w}{\partial \eta^2}\right)+\frac{6}{5(1-\nu)a}\left(\frac{\partial V_x}{\partial \xi}+\nu\frac{\partial V_y}{\partial \eta}\right)+$$
$$+\frac{3\nu}{5(1-\nu)}p-\left\{\frac{1}{70}\left(\frac{h}{a}\right)^2\left(\frac{\partial A_{3x}}{\partial \xi}+\nu\frac{\partial A_{3y}}{\partial \eta}\right)-\frac{\nu}{140(1-\nu)}\left(\frac{h}{a}\right)^2 B_3\right\} \quad (xy) \qquad (1.9)$$

$$\frac{6M_{xy}}{h^2}=-\frac{Eh}{2(1+\nu)a^2}\frac{\partial^2 w}{\partial \xi\,\partial \eta}+\frac{3}{5a}\left(\frac{\partial V_x}{\partial \eta}+\frac{\partial V_y}{\partial \xi}\right)-\left\{\frac{1}{140}\left(\frac{h}{a}\right)^2\left(\frac{\partial A_{3x}}{\partial \eta}+\frac{\partial A_{3y}}{\partial \xi}\right)\right\}$$

$$\sigma_{3xx}=-\frac{\nu}{10(1-\nu)}p-\frac{1}{5(1-\nu)a}\left(\frac{\partial V_x}{\partial \xi}+\nu\frac{\partial V_y}{\partial \eta}\right)+$$
$$+\frac{\nu}{10a^2}\left[\frac{\partial^2(M_{xx}+M_{yy})}{\partial \xi^2}+\nu\frac{\partial^2(M_{xx}+M_{yy})}{\partial \eta^2}\right]-$$
$$-\frac{1}{90(1-\nu^2)}\left(\frac{h}{a}\right)^2\left(\frac{\partial^2 p}{\partial \xi^2}+\nu\frac{\partial^2 p}{\partial \eta^2}\right)+\frac{1}{45(1-\nu)}\left(\frac{h}{a}\right)^2\left(\frac{\partial A_{3x}}{\partial \xi}+\nu\frac{\partial A_{3y}}{\partial \eta}\right)-$$
$$-\frac{\nu}{90(1-\nu)}\left(\frac{h}{a}\right)^2 B_3-\frac{\nu}{90(1-\nu^2)}\left(\frac{h}{a}\right)^4\left[\frac{\partial^2(\sigma_{3xx}+\sigma_{3yy})}{\partial \xi^2}+\right.$$
$$\left.+\nu\frac{\partial^2(\sigma_{3xx}+\sigma_{3yy})}{\partial \eta^2}\right]-\frac{1}{7920(1-\nu^2)}\left(\frac{h}{a}\right)^2\left(\frac{\partial^2 B_3}{\partial \xi^2}+\nu\frac{\partial^2 B_3}{\partial \eta^2}\right) \quad (xy) \qquad (1.10)$$

$$\sigma_{3xy}=-\frac{1}{10a}\left(\frac{\partial V_x}{\partial \eta}+\frac{\partial V_y}{\partial \xi}\right)+\frac{\nu}{10(1+\nu)a^2}\frac{\partial^2(M_{xx}+M_{yy})}{\partial \xi\,\partial \eta}-$$
$$-\frac{1}{90(1+\nu)}\left(\frac{h}{a}\right)^2\frac{\partial^2 p}{\partial \xi\,\partial \eta}+\frac{1}{90}\left(\frac{h}{a}\right)^2\left(\frac{\partial A_{3x}}{\partial \eta}+\frac{\partial A_{3y}}{\partial \xi}\right)-$$
$$-\frac{\nu}{90(1+\nu)}\left(\frac{h}{a}\right)^2\frac{\partial^2(\sigma_{3xx}+\sigma_{3yy})}{\partial \xi\,\partial \eta}-\frac{1}{7920(1+\nu)}\left(\frac{h}{a}\right)^4\frac{\partial^2 B_3}{\partial \xi\,\partial \eta}$$

$$\varphi_x=-\frac{1}{a}\frac{\partial w}{\partial \xi}+\frac{12(1+\nu)}{5Eh}V_x-\frac{(1+\nu)}{35E}\frac{h}{a}A_{3x} \quad (xy) \qquad (1.11)$$

In addition, on the contour of the region

$$\oint (u_n \delta T_{nn} + u_s \delta T_{ns} + w \delta V_n + \varphi_n \delta M_{nn} + \varphi_s \delta M_{ns})\, ds +$$
$$+ \oint \sum_{k=2}^{\infty} \{[\ldots]_k \,\delta\sigma_{knn} + [\ldots]_k \,\delta\sigma_{kns} + [\ldots]_k \delta A_{kn}\}\, ds = 0 \qquad (1.12)$$

must hold.

The quantities under the integral sign are components corresponding to vectors and tensors, referred to the external normal n and the tangent s of the contour. Quantities in the square brackets will not be used and we shall not discuss them.

The static and geometric (homogeneous) boundary conditions follow from Equation (1.12).

Note that if only the first two terms in series (1.1) are retained, the plate theory of Reissner [4] is obtained. If, in addition, the stresses σ_{xz}, σ_{yz}, σ_{zz} are neglected in the functional (1.6), the plate theory of Kirchoff is obtained.

By consideration of the obtained equations it is easy to find that the problem of stress determination splits into two independent problems. The first problem consists in solving equations of the type of (1.7), (1.8) and the first of (1.4) with corresponding boundary conditions from (1.12). The second problem consists in solving equations of the type of (1.9), (1.10), (1.11), and the second equation of (1.4) with the corresponding boundary conditions.

In the future we shall limit ourselves to the case where only the first four terms are retained in the series (1.1); (k = 2, 3).

From the obtained system of equations one may derive another system for the determination of the self-equilibrating stresses σ_{kxx}, σ_{kxy}, σ_{kyy} through the thickness of the plate (excluding forces, moments and displacements appearing as Lagrangean multipliers). The coefficients for the derivatives of different series are small numbers, the smaller the higher the order of the derivative. In view of this, a more general solution for such a homogeneous system will be expressed by a rapidly varying function, and a particular solution is easily calculated with sufficient accuracy. It will also be determined during a study of the state of stress in the plate remote from the edge as a rapidly varying part of the solution by virtue of the St. Venant principle (since σ_{knn}, σ_{kns}, A_{kn} at the edge of the plate determine stresses, self-equilibrating through the thickness) localized at the edge of the plate and decaying rapidly away from it.

In this work on stress determination we have limited ourselves to consideration of a particular solution of the above equations not giving the state of stress localized at the edge of the plate. For this, terms of the order of $(h/a)^2$ must be retained in the formulas for the stresses σ_{xx}, σ_{xy}, σ_{yy}, and terms of the order of $(h/a)^4$ neglected by comparison with unity.

2. We consider the second problem. The shear forces and moments arising from the action of the surface forces (1.2) have the following orders:

$$V_x, \quad V_y \approx pa, \quad M_{xx}, \quad M_{xy}, \quad M_{yy} \approx pa^2 \tag{2.1}$$

(assuming that the external loads are such that the order of the quantities considered does not reduce upon differentiation with respect to ξ, η).

The particular solution of Equation (1.10) with an accuracy to terms of the order of $(h/a)^2$ has the following form:

$$\sigma_{3xx} = -\frac{\nu}{10(1-\nu)}p - \frac{1}{5(1-\nu)a}\left(\frac{\partial V_x}{\partial \xi} + \nu\frac{\partial V_y}{\partial \eta}\right) + \frac{\nu}{10a^2}\left[\frac{\partial^2(M_{xx}+M_{yy})}{\partial \xi^2} + \nu\frac{\partial^2(M_{xx}+M_{yy})}{\partial \eta^2}\right] \qquad (xy)$$

$$\sigma_{3xy} = -\frac{1}{10a}\left(\frac{\partial V_x}{\partial \eta} + \frac{\partial V_y}{\partial \xi}\right) + \frac{\nu}{10(1+\nu)a^2}\frac{\partial^2(M_{xx}+M_{yy})}{\partial \xi\,\partial \eta} \tag{2.2}$$

We conclude from these equations, as well as from (2.1) and (1.5), that the bracketed terms in (1.9) are as small by comparison with the left-hand side of (1.9) as $(h/a)^4$ is by comparison with unity, and these terms must accordingly be rejected in the simplified system.

The expressions obtained for moments

$$\frac{6M_{xx}}{h^2} = -\frac{Eh}{2(1-\nu^2)a^2}\left(\frac{\partial^2 w}{\partial \xi^2} + \nu\frac{\partial^2 w}{\partial \eta^2}\right) + \frac{6}{5(1-\nu)a}\left(\frac{\partial V_x}{\partial \xi} + \nu\frac{\partial V_y}{\partial \eta}\right) + \frac{3\nu}{5(1-\nu)}p \qquad (xy) \tag{2.3}$$

$$\frac{6M_{xy}}{h^2} = -\frac{Eh}{2(1+\nu)a^2}\frac{\partial^2 w}{\partial \xi\,\partial \eta} + \frac{3}{5a}\left(\frac{\partial V_x}{\partial \eta} + \frac{\partial V_y}{\partial \xi}\right)$$

coincide with the formulas of Reissner [4] and Ambartsumian [2]. These formulas would be obtained if the last term on the right in (1.11) were neglected. Therefore one must take

$$\varphi_x = -\frac{1}{a}\frac{\partial w}{\partial \xi} + \frac{12(1+\nu)}{5Eh}V_x \qquad (xy) \tag{2.4}$$

The formulation of the edge problem coincides with the edge problem in the work of Reissner [4], but differs from the edge problem in the work of Ambartsumian [2]. Thus, in the work of Reissner the edge condition is equivalent to

$$-\frac{\partial w}{\partial n}+\frac{3(1+\nu)}{Eh}V_n=0 \tag{2.6}$$

and in the work of Ambartsumian to

$$-\frac{\partial w}{\partial n}+\frac{12(1+\nu)}{5Eh}V_n=0 \tag{2.5}$$

Condition (2.5) is obtained by a variational method. The left-hand side has the significance of a generalized displacement, in which work of the moment M_{nn} is done. The left-hand side of (2.6) has no such significance. Therefore the edge condition (2.6) does not answer the requirement that the reaction of the support do no work, and it must be regarded as inconsistent. The result of this will be a nonselfconjugate edge problem. Thus, the edge problem in the present formulation agrees with the edge condition for plate bending of Reissner. The difference consists in the formulas from which stresses are calculated after finding the forces, moments and displacement w. Thus the stresses σ_{xx}, σ_{xy}, σ_{yy} are determined from the formulas

$$\sigma_{xx}=-\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+$$
$$+\frac{3}{1-\nu}\frac{z}{h}\left(1-\frac{4z^2}{3h^2}\right)\left(\frac{\partial V_x}{\partial x}+\nu\frac{\partial V_y}{\partial y}\right)+\frac{\nu}{1-\nu}\frac{3z}{2h}\left(1-\frac{4z^2}{3h^2}\right)p-$$
$$-\frac{\nu Ez^3}{6(1-\nu)}\Delta\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+\frac{\nu Eh^3}{40(1-\nu)}\frac{z}{h}\Delta\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\qquad (xy) \tag{2.7}$$

$$\sigma_{xy}=-\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x\,\partial y}+\frac{3z}{2h}\left(1-\frac{4z^2}{3h^2}\right)\left(\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}\right)-$$
$$-\frac{\nu Ez^3}{6(1-\nu^2)}\frac{\partial^2\Delta w}{\partial x\,\partial y}+\frac{\nu Eh^3}{40(1-\nu^2)}\frac{z}{h}\frac{\partial^2\Delta w}{\partial x\,\partial y}\qquad\left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

in accordance with (1.1), (2.2) and (2.3).

The first terms on the right-hand side give stresses as calculated from the classical theory of plates. The remaining terms give the corrections of the order of $(h/a)^2$ compared with unity.

If terms $P_5(\zeta)\sigma_{5xx}$, $P_5(\zeta)\sigma_{5xy}$, $P_5(\zeta)\sigma_{5yy}$ were retained in the series (1.1), then the corrections obtained would be of a still higher order than $(h/a)^2$.

In the Reissner plate theory the stresses are

$$\sigma_{xx} = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + \frac{12}{5(1-\nu)}\frac{z}{h}\left(\frac{\partial V_x}{\partial x} + \nu \frac{\partial V_y}{\partial y}\right) + \frac{6\nu}{5(1-\nu)}\frac{z}{h}p$$

$$\sigma_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y} + \frac{6}{5}\frac{z}{h}\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) \qquad (xy) \qquad (2.8)$$

We present for comparison the formulas of Ambartsumian [2]

$$\sigma_{xx} = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + $$
$$+ \frac{3}{1-\nu}\frac{z}{h}\left(1 - \frac{4z^2}{3h^2}\right)\left(\frac{\partial V_x}{\partial x} + \nu \frac{\partial V_y}{\partial y}\right) + \frac{\nu}{1-\nu}\frac{3z}{2h}\left(1 - \frac{4z^2}{3h^2}\right)p \qquad (xy)$$
$$\sigma_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y} + \frac{3z}{2h}\left(1 - \frac{4z^2}{3h^2}\right)\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) \qquad (2.9)$$

and the formulas obtained by Mushtari [3]

$$\sigma_{xx} = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + \frac{3}{1-\nu}\frac{z}{h}\left(1 - \frac{4z^2}{3h^2}\right)\left(\frac{\partial V_x}{\partial x} + \nu \frac{\partial V_y}{\partial y}\right) +$$
$$+ \frac{\nu}{1-\nu}\frac{3z}{2h}\left(1 - \frac{4z^2}{3h^2}\right)p - \frac{\nu E z^3}{6(1-\nu)(1-\nu^2)}\Delta\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \qquad (xy)$$
$$\sigma_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y} + \frac{3z}{2h}\left(1 - \frac{4z^2}{3h^2}\right)\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) - \frac{\nu E z^3}{6(1-\nu^2)}\frac{\partial^2 \Delta w}{\partial x \partial y} \qquad (2.10)$$

A comparison of Formulas (2.8) to (2.10) with those of (2.7) shows that certain terms are missing from the less exact formulas, terms having in general the same order as those already retained.

The reason for this is that Reissner starts with a linear distribution of the stresses σ_{xx}, σ_{xy}, σ_{yy} through the thickness; other authors [2,3] introduced other simplifying assumptions with an unknown error in the determination of stresses and displacements.

3. We return to the problem of the momentless deformation of a plate. We assume that the stresses induced by forces applied at the edge of the plate all have the same order as the bending stresses (i.e. $p(a/h)^2$); let $q \approx p$.

The particular solution of Equation (1.8) with accuracy to terms in $(h/a)^2$ compared with unity will be

$$\sigma_{2xx} = \frac{\nu h}{12(1-\nu^2)a^2}\left[\frac{\partial^2 (T_{xx} + T_{yy})}{\partial \xi^2} + \nu \frac{\partial^2 (T_{xx} + T_{yy})}{\partial \eta^2}\right]$$

$$\sigma_{2xy} = \frac{\nu h}{12(1+\nu)a^2}\frac{\partial^2 (T_{xx} + T_{yy})}{\partial \xi \partial \eta} \qquad (xy) \qquad (3.1)$$

Considering that T_{xx}, $T_{yy} \approx ph(a/h)^2$ and referring to (1.5), we conclude that the bracketed terms in (1.7) are as small in comparison with the left-hand part of (1.7) as $(h/a)^4$ is compared with unity, and accordingly those terms must be omitted in the simplified system. We obtain

$$\frac{T_{xx}}{h} = \frac{E}{(1-\nu^2)\,a}\left(\frac{\partial u}{\partial \xi} + \nu\frac{\partial v}{\partial \eta}\right) + \frac{\nu}{2(1-\nu)}\,q \quad (xy)$$
$$\frac{T_{xy}}{h} = \frac{E}{2(1+\nu)\,a}\left(\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi}\right) \tag{3.2}$$

These formulas are also found in [2,3] in which they determine the membrane stresses. In the present work the stresses are given by the formulas

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) + \frac{\nu}{2(1-\nu)}\,q +$$
$$+\frac{\nu E h^2}{24(1-\nu)(1-\nu^2)}\left(\frac{12z^2}{h^2} - 1\right)\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] \quad (xy)$$
$$\sigma_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\nu E h^2}{24(1-\nu^2)}\left(\frac{12z^2}{h^2} - 1\right)\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

in agreement with (3.1) and (1.1).

If the terms $P_4(\zeta)\sigma_{4xx}$, $P_4(\zeta)\sigma_{4xy}$, $P_4(\zeta)\sigma_{4yy}$ are retained in the series (1.1), then the corrections obtained will be of a higher order of smallness than $(h/a)^2$ as compared to unity.

BIBLIOGRAPHY

1. Cicala, Placido, Sulla teoria della parete sotile. *Giorn. genio civile* 97, Nos. 4, 6, 9, 1959.

2. Ambartsumian, S.A., *Teoriia anizotropnykh obolochek (Theory of Anisotropic Shells)*. Fizmatgiz, 1961.

3. Mushtari, Kh.M., Teoriia izgiba plit srednei tolshchiny (Theory of bending of plates of medium thickness). *Izv. Akad. Nauk SSSR, Mekhanika i Mashinostroenie* No. 2, 1959.

4. Reissner, E., On the bending of elastic plates. *J. Math. and Phys.* Vol. 23, 1944.

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